Forbidden subgraphs for connectivity and supereulerian properties of graphs

Shipeng Wang

Beijing Institute of Technology

7th March, Plzen,Czech

Joint work with Liming Xiong

Outline

1. Forbidden pairs for equality of vertex-connectivity and edge-connectivit of graphs
2. Forbidden subgraph, Independent number for supereulerianity of 3-edge-connected claw-free graphs
3. Problems

Terminology

All graphs are finite, simple and undirected.

$\alpha(G)$: the independence number of G,

$\alpha'(G)$: the matching number of G,

$\kappa(G)$: the connectivity of G,

$\kappa'(G)$: the edge connectivity of G.

Terminology

A graph G is *hamiltonian*: G has a spanning cycle.

A graph G is *supereulerian*: G has a spanning even subgraph.

A *dominating subgraph* D of H is a subgraph such that every edge of H has at least one end vertex contained in D.

Terminology

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be *$\mathcal{H}$-free* if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph*. We call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}| = 2$.

For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there exists a graph H_1 in $\mathcal{H}_1$ such that H_1 is an induced subgraph of H_2.

Remark

If $\mathcal{H}_1 \preceq \mathcal{H}_2$, then $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Terminology

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be *$\mathcal{H}$-free* if G does not contain H as an induced subgraph for all H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph*. We call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}| = 2$.

For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph H_2 in $\mathcal{H}_2$, there exists a graph H_1 in $\mathcal{H}_1$ such that H_1 is an induced subgraph of H_2.

Remark

If $\mathcal{H}_1 \preceq \mathcal{H}_2$, then $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

Outline

1. Forbidden pairs for equality of vertex-connectivity and edge-connectivit of graphs

2. Forbidden subgraph, Independent number for supereulerianity of 3-edge-connected claw-free graphs

3. Problems

Motivation

Theorem 1.1 (Bedrossian, 1991)

Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected graph $\mathcal{H}$-free graph G is hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_6\}, \mathcal{H} \preceq \{K_{1,3}, B_{1,2}\}$, or $\mathcal{H} \preceq \{K_{1,3}, N_{1,1,1}\}$.
[a]

---

[a] Bedrossian, Forbidden subgraph and minimum degree conditions for Hamiltonicity, Ph.D. Thesis, Memphis State University, 1991

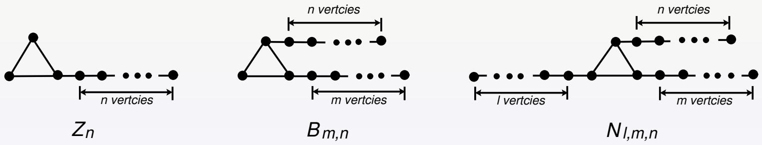

Figure: $Z_n, B_{m,n}$ and $N_{l,m,n}$

Motivation

Because some non-Hamiltonian 2-connected graph with small order, Faudree and Gould have considered forbidden pairs of 2-connected graphs with large order.

Theorem 1.2 (Faudree and Gould, DM, 1997)

Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected graph $\mathcal{H}$-free graph with sufficiently large order is hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_6\}, \mathcal{H} \preceq \{K_{1,3}, B_{1,2}\}, \mathcal{H} \preceq \{K_{1,3}, N_{1,1,1}\}$, or $\mathcal{H} \preceq \{K_{1,3}, Z_3\}$. [a]

[a] Faudree and Gould, Characterizing forbidden pairs for Hamiltonian properties, Discrete Math. 173(1997)45-60

Motivation

Because some non-Hamiltonian 2-connected graph with small order, Faudree and Gould have considered forbidden pairs of 2-connected graphs with large order.

Theorem 1.2 (Faudree and Gould, DM, 1997)

Let $\mathcal{H}$ be a forbidden pair. Then every 2-connected graph $\mathcal{H}$-free graph with sufficiently large order is hamiltonian if and only if $\mathcal{H} \preceq \{K_{1,3}, P_6\}, \mathcal{H} \preceq \{K_{1,3}, B_{1,2}\}, \mathcal{H} \preceq \{K_{1,3}, N_{1,1,1}\}$, or $\mathcal{H} \preceq \{K_{1,3}, Z_3\}$. [a]

[a] Faudree and Gould, Characterizing forbidden pairs for Hamiltonian properties, Discrete Math. 173(1997)45-60

Motivation

It is well-known that every cubic graph G satisfies $\kappa(G) = \kappa'(G)$, we consider the following problem.

Problem: Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every connected $\mathcal{H}$-free graph G with sufficiently larger order satisfies $\kappa(G) = \kappa'(G)$.

Our results

Concerning this problem, set $\mathcal{H}_1 = \{Z_1, P_5\}$, $\mathcal{H}_2 = \{Z_1, K_{1,4}\}$, $\mathcal{H}_3 = \{Z_1$, chair $\}$, $\mathcal{H}_4 = \{P_4$, hourglass$\}$, $\mathcal{H}_5 = \{K_{1,3}$, hourglass$\}$.

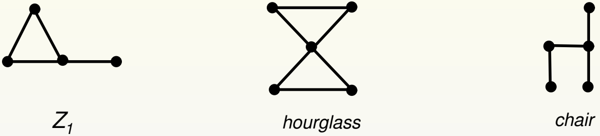

Figure: Z_1, hourglass,chair

Theorem 1.3 (Wang and Xiong, 2016^+)

Let $\mathcal{H}$ be a forbidden pair. If for arbitrary n_0 such that every connected $\mathcal{H}$-free graph G with order at least n_0 satisfies $\kappa(G) = \kappa'(G)$, then $\mathcal{H} \preceq \mathcal{H}_i$ for some i with $1 \leq i \leq 5$ (necessity).

Our results

Concerning this problem, set $\mathcal{H}_1 = \{Z_1, P_5\}$, $\mathcal{H}_2 = \{Z_1, K_{1,4}\}$, $\mathcal{H}_3 = \{Z_1, \text{chair}\}$, $\mathcal{H}_4 = \{P_4, \text{hourglass}\}$, $\mathcal{H}_5 = \{K_{1,3}, \text{hourglass}\}$.

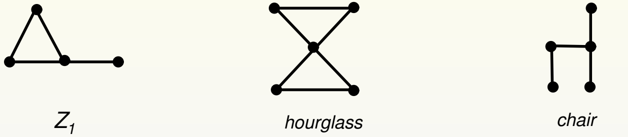

Figure: Z_1, hourglass,chair

Theorem 1.3 (Wang and Xiong, 2016^+)

Let $\mathcal{H}$ be a forbidden pair. If for arbitrary n_0 such that every connected $\mathcal{H}$-free graph G with order at least n_0 satisfies $\kappa(G) = \kappa'(G)$, then $\mathcal{H} \preceq \mathcal{H}_i$ for some i with $1 \leq i \leq 5$ (necessity).

Our results

Theorem 1.4 (Wang and Xiong, 2016^+)

Let $\mathcal{H}$ be a forbidden pair. If $\mathcal{H} \in \{\mathcal{H}_i : 1 \leq i \leq 4\}$, then every connected $\mathcal{H}$-free graph G satisfies $\kappa(G) = \kappa'(G)$. (sufficiency).

Conjecture 1.5

Let G be a connected $\{K_{1,3}$, hourglass$\}$-free graph. Then $\kappa(G) = \kappa'(G)$.

If Conjecture 1.5 is true, then the reverse side of Theorem 1,3 is true.

If Conjecture 1.5 is not true, then since any proper induced subgraph of hourglass is also an induced subgraph of Z_1 and $K_{1,3}$ is an induced subgraph of $K_{1,4}$, the reverse side of Theorem 1,4 is true following Theorems 1.3 and 1.4.

Our results

Theorem 1.4 (Wang and Xiong, 2016^+)

Let $\mathcal{H}$ be a forbidden pair. If $\mathcal{H} \in \{\mathcal{H}_i : 1 \le i \le 4\}$, then every connected $\mathcal{H}$-free graph G satisfies $\kappa(G) = \kappa'(G)$. (sufficiency).

Conjecture 1.5

Let G be a connected $\{K_{1,3}$, hourglass$\}$-free graph. Then $\kappa(G) = \kappa'(G)$.

If Conjecture 1.5 is true, then the reverse side of Theorem 1,3 is true.

If Conjecture 1.5 is not true, then since any proper induced subgraph of hourglass is also an induced subgraph of Z_1 and $K_{1,3}$ is an induced subgraph of $K_{1,4}$, the reverse side of Theorem 1,4 is true following Theorems 1.3 and 1.4.

Our results

We almost established a full characterization, leaving $\mathcal{H} = \{K_{1,3}$, hourglass$\}$ as the only open case and we also partially confirm it, when G is a k-edge-connected graph with $k \leq 3$.

Theorem 1.6 (Wang and Xiong, 2016^+)

Let G be a k-edge-connected $\{K_{1,3}$, hourglass$\}$-free graph with $1 \leq k \leq 3$. Then $\kappa(G) = \kappa'(G)$.

Conjecture 1.5 is still open when G is k-edge-connected with $k \geq 4$. Next we can give an application of Theorem 1.6.

Our results

We almost established a full characterization, leaving $\mathcal{H} = \{K_{1,3}$, hourglass$\}$ as the only open case and we also partially confirm it, when G is a k-edge-connected graph with $k \leq 3$.

Theorem 1.6 (Wang and Xiong, 2016^{+})

Let G be a k-edge-connected $\{K_{1,3}$, hourglass$\}$-free graph with $1 \leq k \leq 3$. Then $\kappa(G) = \kappa'(G)$.

Conjecture 1.5 is still open when G is k-edge-connected with $k \geq 4$. Next we can give an application of Theorem 1.6.

An application of Theorem 1.6

Theorem 1.7 (Ryjáček, Vrána and Xiong, 2016^{+})

Let G be a 3-connected $\{K_{1,3}$, hourglass$\}$-free graph. Then

- if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i+j+k \leq 9$ $(i,j,k \geq 1)$, then G is hamiltonian,
- if G is P_{12}-free, then G is Hamilton-connected.

By Theorem 1.6, the 3-connected condition of Theorem 1.7 can be weakened 3-edge-connected.

Theorem 1.8 (Wang and Xiong, 2016^{+})

Let G be a 3-edge-connected $\{K_{1,3}$, hourglass$\}$-free graph. Then

- if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i+j+k \leq 9$ $(i,j,k \geq 1)$, then G is hamiltonian,
- if G is P_{12}-free, then G is Hamilton-connected.

An application of Theorem 1.6

Theorem 1.7 (Ryjáček, Vrána and Xiong, 2016^+)

Let G be a 3-connected $\{K_{1,3}$, hourglass$\}$-free graph. Then

- if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i+j+k \leq 9$ $(i,j,k \geq 1)$, then G is hamiltonian,
- if G is P_{12}-free, then G is Hamilton-connected.

By Theorem 1.6, the 3-connected condition of Theorem 1.7 can be weakened 3-edge-connected.

Theorem 1.8 (Wang and Xiong, 2016^+)

Let G be a 3-edge-connected $\{K_{1,3}$, hourglass$\}$-free graph. Then

- if G is P_{20}-free, Z_{18}-free, or $N_{2i,2j,2k}$-free with $i+j+k \leq 9$ $(i,j,k \geq 1)$, then G is hamiltonian,
- if G is P_{12}-free, then G is Hamilton-connected.

Sketch the proof of Theorem 1.6

- Suppose, by contradiction, that $\kappa(G) < \kappa'(G)$. Then, since G is k-edge-connected with $1 \le k \le 3$, we have $1 \le \kappa(G) < \kappa'(G) \le 3$. If $\kappa(G) = 1$ and $\kappa'(G) \ge 2$, then G has an induced $K_{1,3}$ or a hourglass, a contradiction.
- Hence we have $\kappa(G) = 2$ and $\kappa'(G) = 3$. Let $S := \{v_1, v_2\}$ be a minimum vertex-cut of G. Then by the minimality of S, each v_i has at least one neighbor in every component of $G - S$. Therefore, since G is claw-free, $G - S$ has exactly two components D_1 and D_2.

Sketch the proof of Theorem 1.6

- Suppose, by contradiction, that $\kappa(G) < \kappa'(G)$. Then, since G is k-edge-connected with $1 \leq k \leq 3$, we have $1 \leq \kappa(G) < \kappa'(G) \leq 3$. If $\kappa(G) = 1$ and $\kappa'(G) \geq 2$, then G has an induced $K_{1,3}$ or a hourglass, a contradiction.
- Hence we have $\kappa(G) = 2$ and $\kappa'(G) = 3$. Let $S := \{v_1, v_2\}$ be a minimum vertex-cut of G. Then by the minimality of S, each v_i has at least one neighbor in every component of $G - S$. Therefore, since G is claw-free, $G - S$ has exactly two components D_1 and D_2.

- By some arguments and discussion we can get a series of such vertices: x_i, y_i (see Figure). Let $X = \bigcup_{i=1}^{l}\{x_i\}, Y = \bigcup_{i=1}^{l}\{y_i\}$. In fact, we have $V(D_1) = X \cup Y$, and we claim the following.
- **Claim**: $G[\{x_i, y_i, x_{i+1}, y_{i+1}\}]$ is a $K_4 \backslash \{e\}$.
- The above claim implies that i may be any large, but this contradicts the fact that G is finite. This shows that $\kappa(G) = \kappa'(G)$

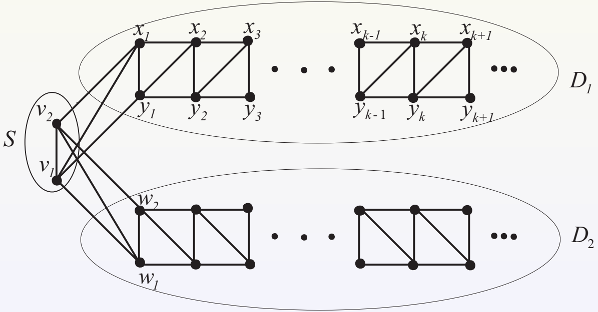

Figure: Illustration in **Claim**

Outline

Motivation

Theorem 2.1 (Łuczak and Pfender, JGT, 2004)

Every 3-connected $\{$claw, $P_{11}\}$-free graph is hamiltonian.
[a]

[a] T. Łuczak and F. Pfender, Claw-free 3-connected P_{11}-free graphs are Hamiltonian, J. Graph Theory, 47(2004)111-121.

Question A. What is the largest k such that every 3-edge-connected claw-free and P_k-free graph is supereulerian?

Motivation

Theorem 2.1 (Łuczak and Pfender, JGT, 2004)

Every 3-connected $\{$claw, $P_{11}\}$-free graph is hamiltonian.
[a]

[a] T. Łuczak and F. Pfender, Claw-free 3-connected P_{11}-free graphs are Hamiltonian, J. Graph Theory, 47(2004)111-121.

Question A. What is the largest k such that every 3-edge-connected claw-free and P_k-free graph is supereulerian?

Motivation

Theorem 2.2 (Chen, GC, 2016)

Let H be a 3-connected claw-free graph with $\alpha(H) \leq 7$ is either hamiltonian or $cl(H) = L(G)$ where G is a graph with $\alpha'(G) = 7$ that is obtained from the Petersen graph P by adding some pendant edges or subdividing some edges of P.

Theorem 2.3 (An and Xiong, AMAS, 2015)

Let G be a 3-edge-connected graph with independent number $\alpha'(G) \leq 5$. Then G is supereulerian if and only if G can not be contracted to Petersen graph P.

Question B. What is the largest k such that every 3-edge-connected claw-free graph G with independent number $\alpha(G) \leq k$ is supereulerian?

Motivation

Theorem 2.2 (Chen, GC, 2016)

Let H be a 3-connected claw-free graph with $\alpha(H) \leq 7$ is either hamiltonian or $cl(H) = L(G)$ where G is a graph with $\alpha'(G) = 7$ that is obtained from the Petersen graph P by adding some pendant edges or subdividing some edges of P.

Theorem 2.3 (An and Xiong, AMAS, 2015)

Let G be a 3-edge-connected graph with independent number $\alpha'(G) \leq 5$. Then G is supereulerian if and only if G can not be contracted to Petersen graph P.

Question B. What is the largest k such that every 3-edge-connected claw-free graph G with independent number $\alpha(G) \leq k$ is supereulerian?

Motivation

Theorem 2.4 (Chen, Lai and Zhang, DM, 2017)

Let G be a 3-edge-connected graph with matching number $\alpha'(G) \leq 7$. Then G is supereulerian if and only if G can not be contracted Petersen graph P and P^*. [a]

[a] Z.-H. Chen, H.-J. Lai and M. Zhang, Spanning trails with variations of Chvátal-Erdös Type Conditions, Discrete Math., 340(2017)243-251.

Question C. What is the largest k such that every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq k$ is supereulerian?

We can answer Questions A and B but we can not answer Question C.

Motivation

Theorem 2.4 (Chen, Lai and Zhang, DM, 2017)

Let G be a 3-edge-connected graph with matching number $\alpha'(G) \leq 7$. Then G is supereulerian if and only if G can not be contracted Petersen graph P and P^*.
[a]

[a] Z.-H. Chen, H.-J. Lai and M. Zhang, Spanning trails with variations of Chv$\acute{a}$tal-Erdös Type Conditions, Discrete Math., 340(2017)243-251.

Question C. What is the largest k such that every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq k$ is supereulerian?

We can answer Questions A and B but we can not answer Question C.

Our results

Theorem 2.5 (Wang and Xiong, 2016^+)(sharpness)

Every 3-edge-connected $\{$claw, $P_{20}\}$-free graph is supereulerian.

Because independent number $\alpha(G) \leq 9$ (respectively, matching number $\alpha'(G) \leq 9$) implies P_{20}-free, we can immediately obtain the following two results.

Corollary 2.6 (Wang and Xiong, 2016^+)(sharpness)

Every 3-edge-connected claw-free graph G with independent number $\alpha(G) \leq 9$ is supereulerian.

Corollary 2.7 (Wang and Xiong, 2016^+)

Every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq 9$ is supereulerian.

Our results

Theorem 2.5 (Wang and Xiong, 2016^+)(sharpness)

Every 3-edge-connected $\{$claw, $P_{20}\}$-free graph is supereulerian.

Because independent number $\alpha(G) \leq 9$ (respectively, matching number $\alpha'(G) \leq 9$) implies P_{20}-free, we can immediately obtain the following two results.

Corollary 2.6 (Wang and Xiong, 2016^+)(sharpness)

Every 3-edge-connected claw-free graph G with independent number $\alpha(G) \leq 9$ is supereulerian.

Corollary 2.7 (Wang and Xiong, 2016^+)

Every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq 9$ is supereulerian.

Illustration of sharpness of Theorem 2.5 and Corollary 2.6

Now we will show the line graph of the subdivision H of the Petersen graph is non-supereulerian and then we can show Theorem 2.5 and Corollary 2.6 are sharp, and Corollary 2.7 may not be sharp.

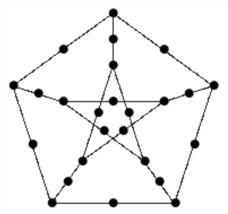

Figure: Subdivision H of the Petersen graph

Illustration of sharpness of Theorem 2.5 and Corollary 2.6

Theorem 2.8 (Xiong, DM, 2014)

Let H be a graph with $|E(H)| \geq 3$. Then $L(H)$ is supereulerian if and only if H has a dominating subgraph D satisfying the following conditions:

(i) D is an even graph,

(ii) Each isolated vertex (trivial component) of H has degree at least three in H,

(iii) $d_H(F, D-F) = 1$ for any subgraph $F \subseteq D$.

L. Xiong, Induced hourglass and the equivalence between hamiltonicity and supereulerianity in claw-free graphs, Discrete Math. 332 (2014), 15-22.

Because the subdivision H of the Petersen graph has no dominating subgraph satisfies Conditions $(i), (ii), (iii)$ of Theorem 2.8, $L(H)$ is non-supereulerian.

Illustration of sharpness of Theorem 2.5 and Corollary 2.6

The largest path of subdivision H of the Petersen graph is P_{21}, so the largest induced path of $L(H)$ is P_{20} and then Theorem 2.5 is sharp.

Also note that the matching number of subdivision H of the Petersen graph is 10, the independent number of $L(H)$ is 10 and then Corollary 2.6 is sharp.

But the matching number of $L(H)$ is 15 and then Corollary 2.7 may be not sharp.

Conjecture 2.9

Every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq 14$ is supereulerian.

Illustration of sharpness of Theorem 2.5 and Corollary 2.6

The largest path of subdivision H of the Petersen graph is P_{21}, so the largest induced path of $L(H)$ is P_{20} and then Theorem 2.5 is sharp.

Also note that the matching number of subdivision H of the Petersen graph is 10, the independent number of $L(H)$ is 10 and then Corollary 2.6 is sharp.

But the matching number of $L(H)$ is 15 and then Corollary 2.7 may be not sharp.

Conjecture 2.9

Every 3-edge-connected claw-free graph G with matching number $\alpha'(G) \leq 14$ is supereulerian.

The proof of Theorem 2.5

It is well-know that Ryjacek closure $cl(G)$ which preserve hamiltonicity and traceablity of claw-free graphs. Later, Xiong has derived a new closure $cl^{se}(G)$ which strengthen the closure $cl(G)$ of G preserving supereulerianity of claw-free graphs.

- A vertex $x \in V(G)$ is *locally connected* if the neighborhood of x induces a connected subgraph in G. We say a vertex x is *eligible* if x is locally connected and $G[N_G(x)]$ is non-complete.
- The set of eligible vertices of a graph G denoted by $EL(G)$.
- For a vertex $v \in V(G)$, the graph G_x^* with $V(G_x^*) = V(G)$ and $E(G_x^*) = E(G) \cup \{uv | u, v \in N_G(x)\}$ is called the *local completion* of G at x.

The proof of Theorem 2.5

It is well-know that Ryjacek closure $cl(G)$ which preserve hamiltonicity and traceablity of claw-free graphs. Later, Xiong has derived a new closure $cl^{se}(G)$ which strengthen the closure $cl(G)$ of G preserving supereulerianity of claw-free graphs.

- A vertex $x \in V(G)$ is *locally connected* if the neighborhood of x induces a connected subgraph in G. We say a vertex x is *eligible* if x is locally connected and $G[N_G(x)]$ is non-complete.
- The set of eligible vertices of a graph G denoted by $EL(G)$.
- For a vertex $v \in V(G)$, the graph G_x^* with $V(G_x^*) = V(G)$ and $E(G_x^*) = E(G) \cup \{uv|u, v \in N_G(x)\}$ is called the *local completion* of G at x.

The proof of Theorem 2.5

- We denote the largest order of a clique containing e by $\omega_G(e)$.
- Let C_k be a cycle of even length $k \geq 4$. Two edges $e_1, e_2 \in V(C_k)$ are said to be edge-antipodal in C_k if they are at maximum distance in C_k (i.e. $d_{C_k}(e_1, e_2) = \frac{k}{2} - 1$). An even cycle C_k is said to be edge-antipodal, abbreviated EA, if $min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C_k)$.

Let $EL^{se}(G)$ denote the set of vertices satisfying one of the following:

- $x \in EL(G)$,
- x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6,
- x is the center of an induced hourglass (i.e., the unique connected simple graph with a degree sequence 2,2,2,2,4).

The proof of Theorem 2.5

- We denote the largest order of a clique containing e by $\omega_G(e)$.
- Let C_k be a cycle of even length $k \geq 4$. Two edges $e_1, e_2 \in V(C_k)$ are said to be edge-antipodal in C_k if they are at maximum distance in C_k (i.e. $d_{C_k}(e_1, e_2) = \frac{k}{2} - 1$). An even cycle C_k is said to be edge-antipodal, abbreviated EA, if $min\{\omega_G(e_1), \omega_G(e_2)\} = 2$ for any two antipodal edges $e_1, e_2 \in E(C_k)$.

Let $EL^{se}(G)$ denote the set of vertices satisfying one of the following:

- $x \in EL(G)$,
- x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6,
- x is the center of an induced hourglass (i.e., the unique connected simple graph with a degree sequence 2,2,2,2,4).

The proof of Theorem 2.5

A graph $cl^{se}(G)$ is called strengthen closure of a claw-free graph G, if there is a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in EL^{se}(G_i), i = 1, 2, \ldots, k-1$,
- $G_k = cl^{se}(G)$ and $EL^{se}(G) = \emptyset$.

Remark

Let G be a claw-free graph. Then $cl^{se}(G)$ is $\{C_4, C_5,$ hourglass$\}$-free and it has no induced EA-cycle of length 6.

Theorem 2.10 (Xiong, 2011)

Let G be a claw-free graph. Then $cl^{se}(G)$ is supereulerian if and only if G is supereulerian.
[a]

[a] L. Xiong, Closure operation for even factors on claw-free graphs, Discrete Math., 311(2011)1714-1723

The proof of Theorem 2.5

A graph $cl^{se}(G)$ is called strengthen closure of a claw-free graph G, if there is a sequence of graphs $G_1, \ldots, G_k$ such that

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in EL^{se}(G_i), i = 1, 2, \ldots, k-1$,
- $G_k = cl^{se}(G)$ and $EL^{se}(G) = \emptyset$.

Remark

Let G be a claw-free graph. Then $cl^{se}(G)$ is $\{C_4, C_5,$ hourglass$\}$-free and it has no induced EA-cycle of length 6.

Theorem 2.10 (Xiong, 2011)

Let G be a claw-free graph. Then $cl^{se}(G)$ is supereulerian if and only if G is supereulerian.
[a]

[a] L. Xiong, Closure operation for even factors on claw-free graphs, Discrete Math., 311(2011)1714-1723

The proof of Theorem 2.5

Theorem 2.11 (Wang and Xiong, 2016^{+})

Let G be a {claw, P_t}-free graph $(t \geq 1)$. Then the closure $cl^{se}(G)$ is also {claw, P_t}-free.

Theorem 2.12 (Xiong, 2014, DM)

Let G be a {claw, hourglass}-free graph. Then G is hamiltonian if and only if G is supereulerian.

Corollary 2.13 (Wang and Xiong, 2016^{+}) (last section).

Let G be a 3-edge-connected {claw, hourglass, P_{20}}-free graph. Then G is hamiltonian.

Theorem 2.5 follows Remark, Theorems 2.10,2.11,2.12 and Corollary 2.13.

The proof of Theorem 2.5

Theorem 2.11 (Wang and Xiong, 2016^+)

Let G be a {claw, P_t}-free graph ($t \geq 1$). Then the closure $cl^{se}(G)$ is also {claw, P_t}-free.

Theorem 2.12 (Xiong, 2014, DM)

Let G be a {claw, hourglass}-free graph. Then G is hamiltonian if and only if G is supereulerian.

Corollary 2.13 (Wang and Xiong, 2016^+) (last section).

Let G be a 3-edge-connected {claw, hourglass, P_{20}}-free graph. Then G is hamiltonian.

Theorem 2.5 follows Remark, Theorems 2.10,2.11,2.12 and Corollary 2.13.

The proof of Theorem 2.5

Theorem 2.11 (Wang and Xiong, 2016^{+})

Let G be a $\{$claw, $P_t\}$-free graph $(t \geq 1)$. Then the closure $cl^{se}(G)$ is also $\{$claw, $P_t\}$-free.

Theorem 2.12 (Xiong, 2014, DM)

Let G be a $\{$claw, hourglass$\}$-free graph. Then G is hamiltonian if and only if G is supereulerian.

Corollary 2.13 (Wang and Xiong, 2016^{+}) (last section).

Let G be a 3-edge-connected $\{$claw, hourglass, $P_{20}\}$-free graph. Then G is hamiltonian.

Theorem 2.5 follows Remark, Theorems 2.10,2.11,2.12 and Corollary 2.13.

Problems

How about the forbidden pairs $\{K_{1,3}, Z_k\}, \{K_{1,3}, N_{i,j,k}\}(i, j, k \geq 1)$ for supereulerian property of 3-edge-connected graphs?

Because we already know that every 3-edge-connected {claw,hourglass, Z_{18}}-free, or {claw,hourglass, $N_{2i,2j,2k}$}-free graph with $i + j + k \leq 9$ $(i, j, k \geq 1)$ is hamiltonian, the proof of of Theorem 2.5 indicates that we should verify whether the strength closure $cl^{se}(G)$ preserves the $Z_k, N_{i,j,k}(i, j, k \geq 1)$-free properties of claw-free graphs.

Problems

How about the forbidden pairs $\{K_{1,3}, Z_k\}, \{K_{1,3}, N_{i,j,k}\}(i,j,k \geq 1)$ for supereulerian property of 3-edge-connected graphs?

Because we already know that every 3-edge-connected {claw,hourglass, Z_{18}}-free, or {claw,hourglass, $N_{2i,2j,2k}$}-free graph with $i+j+k \leq 9$ $(i,j,k \geq 1)$ is hamiltonian, the proof of of Theorem 2.5 indicates that we should verify whether the strength closure $cl^{se}(G)$ preserves the $Z_k, N_{i,j,k}(i,j,k \geq 1)$-free properties of claw-free graphs.

Outline

Problems

- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every 3-edge-connected $\mathcal{H}$-free is supereulerian, we already know $\{K_{1,3}, P_{20}\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every $\mathcal{H}$-free G is supereulerian if and only if G is hamiltonian, we already know $\{K_{1,3},$ hourglass $\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every connected $\mathcal{H}$-free G satisfies $\kappa(G) = \kappa'(G)$, we already know leaving the only case $\mathcal{H} = \{K_{1,3},$ hourglass$\}$ open.
- Determine the largest k such that every 3-edge-connected claw-free graph with matching number $\alpha'(G) \leq k$ is supereulerian, we already know $9 \leq k \leq 14$.

Problems

- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every 3-edge-connected $\mathcal{H}$-free is supereulerian, we already know $\{K_{1,3}, P_{20}\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every $\mathcal{H}$-free G is supereulerian if and only if G is hamiltonian, we already know $\{K_{1,3}$, hourglass $\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every connected $\mathcal{H}$-free G satisfies $\kappa(G) = \kappa'(G)$, we already know leaving the only case $\mathcal{H} = \{K_{1,3}$, hourglass$\}$ open.
- Determine the largest k such that every 3-edge-connected claw-free graph with matching number $\alpha'(G) \leq k$ is supereulerian, we already know $9 \leq k \leq 14$.

Problems

- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every 3-edge-connected $\mathcal{H}$-free is supereulerian, we already know $\{K_{1,3}, P_{20}\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every $\mathcal{H}$-free G is supereulerian if and only if G is hamiltonian, we already know $\{K_{1,3}$, hourglass $\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every connected $\mathcal{H}$-free G satisfies $\kappa(G) = \kappa'(G)$, we already know leaving the only case $\mathcal{H} = \{K_{1,3}$, hourglass$\}$ open.
- Determine the largest k such that every 3-edge-connected claw-free graph with matching number $\alpha'(G) \leq k$ is supereulerian, we already know $9 \leq k \leq 14$.

Problems

- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every 3-edge-connected $\mathcal{H}$-free is supereulerian, we already know $\{K_{1,3}, P_{20}\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every $\mathcal{H}$-free G is supereulerian if and only if G is hamiltonian, we already know $\{K_{1,3}$, hourglass $\} \in$ **H**.
- Determine the set **H** of forbidden pairs $\mathcal{H}$ such that every connected $\mathcal{H}$-free G satisfies $\kappa(G) = \kappa'(G)$, we already know leaving the only case $\mathcal{H} = \{K_{1,3}$, hourglass$\}$ open.
- Determine the largest k such that every 3-edge-connected claw-free graph with matching number $\alpha'(G) \leq k$ is supereulerian, we already know $9 \leq k \leq 14$.

Thanks for your attention!